

An existence theorem for the Cauchy problem on a characteristic cone for the Einstein equations

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ABSTRACT. We prove an existence theorem for the Cauchy problem on a characteristic cone for the vacuum Einstein equations.

1. Introduction

In recent work [6] (see also [5, 4]) we have analysed some aspects of the Cauchy problem for the Einstein equations with data on a characteristic cone in all dimensions $n + 1 \geq 3$, see [2, 9, 11, 14] and references therein for previous work on the subject. In this note we apply the results derived in [6] to present an existence theorem for this problem, with initial data which approach rapidly the flat metric near the tip of the light cone, see Theorem 2.1 below.

The reader's attention is drawn to [12], where sets of unconstrained data on a light-cone centered at past timelike infinity are given in dimension $3 + 1$.

2. Free initial data with asymptotic expansions at the vertex

It is well known that by using normal coordinates centred at O the characteristic cone C_O of a given Lorentzian metric can be written, at least in a neighbourhood of O , as a cone in Minkowski spacetime whose generators represent the null rays. It is therefore no geometric restriction to assume that the characteristic cone of the spacetime we are looking for is represented in some coordinates $y := (y^\alpha) \equiv (y^0, y^i, i = 1, \dots, n)$ of \mathbf{R}^{n+1} by the equation of a Minkowskian cone with vertex O ,

$$(2.1) \quad r - y^0 = 0, \quad r := \left\{ \sum (y^i)^2 \right\}^{\frac{1}{2}}.$$

The parameter r is an affine parameter when normal coordinates are used, and it is also going to be an affine parameter in the solutions that we are going to construct.

Coordinates y^μ as above, which moreover satisfy the wave-equation, $\square_g y^\mu = 0$, will be called *normal-wave* coordinates. Given a smooth metric, such coordinates can always be constructed (see [10] or [9]) by solving the wave equation in the

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domain of dependence of C_O , with initial data the normal coordinates on C_O . Given a smooth metric, one obtains a coordinate system near the vertex, which suffices for our purposes. The coordinates y will be normal-wave coordinates for the solution which we aim to construct.

Given the coordinates y^α we can define coordinates x^α on \mathbf{R}^{n+1} by setting

$$(2.2) \quad x^0 = r - y^0, \quad x^1 = r,$$

with x^A local coordinates on S^{n-1} . The null geodesics issued from O have equation $x^0 = 0$, $x^A = \text{constant}$, so that $\frac{\partial}{\partial x^1}$ is tangent to those geodesics. On C_O (but not outside of it in general) the spacetime metric g that we attempt to construct takes the form (we put an overbar to denote restriction to C_O of spacetime quantities)

$$(2.3) \quad \bar{g} := g|_{x^0=0} \equiv \bar{g}_{00}(dx^0)^2 + 2\nu_0 dx^0 dx^1 + 2\nu_A dx^0 dx^A + \tilde{g},$$

where

$$(2.4) \quad \nu_0 := \bar{g}_{01}, \quad \nu_A := \bar{g}_{0A}, \quad \tilde{g} := \bar{g}_{AB} dx^A dx^B, \quad A, B = 2, \dots, n,$$

are respectively an x^1 -dependent scalar, one-form, and Riemannian metric on S^{n-1} . The symbol r will be used interchangeably with x^1 .

To avoid ambiguities, we will write $g_{\mu\nu}$ for the components of a metric tensor in the x^α coordinates, and $\underline{g}_{\mu\nu}$ for the components in the coordinate system y^μ . This convention will be used regardless of whether g is defined on the space-time, or only on the light-cone.

The analysis in [6] uses a wave-map gauge, with Minkowski target \hat{g} , with the light-cone of \hat{g} being the image by the wave-map f of the light-cone of the metric g that one seeks to construct. Quite generally, a metric g on a manifold V will be said to be *in \hat{g} -wave-map gauge* if the identity map $V \rightarrow V$ is a harmonic diffeomorphism from the spacetime (V, g) onto the pseudo-Riemannian manifold (V, \hat{g}) . Recall that a mapping $f : (V, g) \rightarrow (f(V), \hat{g})$ is a harmonic map if it satisfies the equation, in abstract index notation,

$$(2.5) \quad \hat{\square} f^\alpha := g^{\lambda\mu} (\partial_\lambda^2 f^\alpha - \Gamma_{\lambda\mu}^\sigma \partial_\sigma f^\alpha + \partial_\lambda f^\sigma \partial_\mu f^\rho \hat{\Gamma}_{\sigma\rho}^\alpha) = 0.$$

In a subset in which f is the identity map defined by $f^\alpha(y^\mu) = y^\alpha$, the above equation reduces to $H = 0$, where the *wave-gauge vector* H is given in arbitrary coordinates by the formula

$$(2.6) \quad H^\lambda := g^{\alpha\beta} \Gamma_{\alpha\beta}^\lambda - W^\lambda, \quad \text{with } W^\lambda := g^{\alpha\beta} \hat{\Gamma}_{\alpha\beta}^\lambda,$$

where $\hat{\Gamma}_{\alpha\beta}^\lambda$ are the Christoffel symbols of the *target* metric \hat{g} . See [3] for a more complete discussion.

There are various ways of choosing free initial data for the Cauchy problem for the vacuum Einstein equations on the light-cone C_O . In this work we choose as initial data a one-parameter family, parameterized by r , of conformal classes of metrics $[\gamma(r)]$ on S^{n-1} , thus $\tilde{g}(r)$ is assumed to be conformal to $\gamma(r)$, and where r will be an affine parameter in the resulting vacuum space-time. The initial data needed for the evolution equations are the values of the metric tensor on the light-cone, which will be obtained from $\gamma(r)$ by solving a set of wave-map-gauge constraint equations derived in [6], namely Equations (2.13), (2.20), (2.25)-(2.26) and (2.30) below. The main issue is then to understand the behaviour of the fields near the vertex of the light-cone, making sure that the y^α -coordinates components of the metric \bar{g} , obtained by solving the wave-map-gauge constraints, can be written as

restrictions to the light-cone of sufficiently smooth functions on space-time, so that the PDE existence theorem of [9] can be invoked to obtain the vacuum space-time metric.

It is convenient to start with some notation. For $\ell \in \mathbf{N}$ and $\lambda \in \mathbf{R}$ we shall say that a tensor field $\varphi(r, x^2, \dots, x^n)$, of valence N , defined for $0 < r \leq r_0 \leq 1$, is $O_\ell(r^\lambda)$ if there exists a constant C such that

$$\text{for all } j_1 + \dots + j_n \leq \ell \text{ we have } |\partial_r^{j_1} \partial_{x^2}^{j_2} \dots \partial_{x^n}^{j_n} \varphi_{i_1 \dots i_N}| \leq C r^{\lambda - j_1},$$

where $\varphi_{i_1 \dots i_N}$ are coordinate components of φ , in a coordinate system which will be made clear as needed.

Let s_{AB} denote the canonical unit round metric on the sphere S^{n-1} . We have:

Theorem 2.1. *Let $m, n, \ell, N \in \mathbf{N}$, $n \geq 2$, $m > n/2 + 3$, $\ell \geq 2m + 2$, $N \geq 2m + 1$, $\lambda \in [N, N + 1]$. Suppose that there exist smooth tensor fields $\underline{s}_{AB}^{(i)}$ on S^{n-1} , $i = 4, \dots, N$, so that, in local charts on S^{n-1} , the coordinate components γ_{AB} satisfy*

$$(2.7) \quad \gamma_{AB} - r^2 s_{AB} = \sum_{i=4}^N r^i \underline{s}_{AB}^{(i)} + O_\ell(r^\lambda).$$

Then:

(1) *There exist functions $\underline{g}_{\mu\nu}^{(i)} \in C^\infty(S^{n-1})$ such that*

$$\underline{g}_{\mu\nu} - \underline{\eta}_{\mu\nu} = \sum_{i=2}^{N-2} r^i \underline{g}_{\mu\nu}^{(i)} + O_{\ell-3}(r^{\lambda-2}).$$

If there exists $4 \leq N_0 \leq N$ so that $\underline{s}_{AB}^{(i)} = 0$ for $i = 4, \dots, N_0$, then $\underline{g}_{\mu\nu}^{(i)} = 0$ for $i = 2, \dots, N_0 - 2$.

(2) *If moreover*

$$(2.8) \quad \sum_{i=2}^{N-2} r^i \underline{g}_{\mu\nu}^{(i)} \text{ are restrictions to the light-cone } C_O \\ \text{of a polynomial in the variables } y^\mu,$$

then there exists a $C^{m-(n-1)/2}$ Lorentzian metric defined in a neighbourhood of the vertex of C_O , with $\tilde{g}(r) \in [\gamma(r)]$, which is vacuum to the future of O .

Roughly speaking, the index m is the final Sobolev differentiability of the solution. The ranges of indices above are only needed for the second part of the theorem, and arise from the fact that Dossa's existence theorem [9] requires initial data which are of C^{2m-1} differentiability class in coordinates regular near the vertex. There is a loss of three derivatives when going from the free data γ_{AB} to the full initial data $\underline{g}_{\mu\nu}$, which brings the threshold up to $2m + 2$. Finally, the existence argument invokes the Bianchi identity, which in its natural version requires a C^3 metric; together with the Sobolev embedding, this leads to the restriction $m > n/2 + 3$.

Straightforward Taylor expansions at O show that a smooth metric on a space-time (\mathcal{M}, g) will lead to the form (2.7) of $[\gamma(r)]$, with m , ℓ and N which can be chosen at will, and with $\lambda = N + 1$. So (2.7) is necessary in this sense.

When all the $\overset{(i)}{s}_{AB}$'s, $i = 4, \dots, N$, vanish, then so do the $\overset{(i)}{g}_{\mu\nu}$'s. It follows that

Corollary 2.2. *If γ_{AB} approaches $r^2 s_{AB}$ as r^{2m+1} or faster, where m is the smallest integer less than or equal to $n/2 + 3$, then an associated solution of the vacuum Einstein equations exists.*

In dimension three the required rate is $\lambda \geq 11$.

In view of our theorem above, to obtain a complete solution of the problem at hand it remains to provide an exhaustive description of those $[\gamma(r)]$'s that lead to (2.8). We conjecture that (2.8) will hold for all $\gamma(r)$'s arising from the restriction of a smooth metric to a light-cone, where r is an affine parameter. A similar problem arising on the null cone at past infinity in dimensions $3 + 1$ has been solved by Friedrich in [12], but no details have been presented.

PROOF OF THEOREM 2.1: We need to analyze the behaviour of the solutions of the wave-map-gauge constraint equations near the vertex of the cone. We start by noting that, for some smooth functions $\overset{(i)}{\psi}$ on S^{n-1} ,

$$(2.9) \quad \partial_r \gamma_{AB} - 2r s_{AB} = \sum_{i=4}^N i r^{i-1} \overset{(i)}{s}_{AB} + O_{\ell-1}(r^{\lambda-1}),$$

$$(2.10) \quad \frac{1}{2} \gamma^{AB} \partial_r \gamma_{AB} - \frac{n-1}{r} = \sum_{i=1}^{N-3} r^i \overset{(i)}{\psi} + O_{\ell-1}(r^{\lambda-3}),$$

where γ^{AB} is the matrix inverse of γ_{AB} . Further, for some smooth tensors $\overset{(i)}{\sigma}_{AB}$ on S^{n-1} ,

$$(2.11) \quad \gamma_{AC} \overset{(3)}{\sigma}_B^C := \frac{1}{2} \partial_r \gamma_{AB} - \frac{1}{2} \frac{\gamma^{CD} \partial_r \gamma_{CD}}{n-1} \gamma_{AB} = \sum_{i=3}^{N-1} r^i \overset{(i)}{\sigma}_{AB} + O_{\ell-1}(r^{\lambda-1});$$

note that $\overset{(3)}{\sigma}_{AB}$ is the s_{AB} -trace-free part of $\overset{(4)}{s}_{AB}$. This leads to, for some functions $\overset{(i)}{f}$ on S^{n-1} ,

$$(2.12) \quad |\sigma|^2 := \sigma_A^B \sigma_B^A = \sum_{i=2}^{N-2} r^i \overset{(i)}{f} + O_{\ell-1}(r^{\lambda-2}),$$

where σ_A^B has been defined in the left-hand side of (2.11). Let

$$y := \frac{n-1}{\tau},$$

where τ is the divergence of C_O (sometimes denoted by θ in the literature; cf., e.g., [13]):

$$\tau := \frac{1}{2} g^{AB} \partial_r g_{AB}.$$

In terms of y , the vacuum Raychadhuri equation $\overline{R_{11}} \equiv \overline{R_{\mu\nu}} \ell^\mu \ell^\nu = 0$, where ℓ^ν is a null tangent to the generators of C_O , reads

$$(2.13) \quad y' = 1 + \frac{1}{n-1} |\sigma|^2 y^2.$$

Using known arguments (compare [1, 7] and [12, Lemma 8.2]), there exist functions $\overset{(i)}{y} \in C^\infty(S^{n-1})$ such that

$$y = r + \sum_{i=5}^{N+1} r^i \overset{(i)}{y} + O_{\ell-1}(r^{\lambda+1}).$$

We rewrite this as

$$y = r(1 + \delta y), \text{ with } \delta y = \sum_{i=4}^N r^i \overset{(i+1)}{y} + O_{\ell-1}(r^\lambda) = O(r^4),$$

so that

$$\tau = \frac{n-1}{y} = \frac{n-1}{r(1 + \delta y)} = \frac{n-1}{r} \left(1 - \frac{\delta y}{1 + \delta y} \right).$$

Using this last formula, it is easy to establish existence of functions $\overset{(i)}{\tau} \in C^\infty(S^{n-1})$ such that

$$(2.14) \quad \tau - \frac{n-1}{r} = \sum_{i=3}^{N-1} r^i \overset{(i)}{\tau} + O_{\ell-1}(r^{\lambda-1}).$$

Let us write

$$(2.15) \quad \bar{g}_{AB} = e^\omega \gamma_{AB},$$

From

$$(2.16) \quad \tau \equiv \partial_1 \log \sqrt{\det \gamma} + \frac{n-1}{2} \partial_1 \omega,$$

there exist functions $\overset{(i)}{\omega} \in C^\infty(S^{n-1})$ such that:

$$(2.17) \quad \omega = \sum_{i=4}^N r^i \overset{(i)}{\omega} + O_{\ell-1}(r^\lambda).$$

Equation (2.17) implies that there exist smooth tensor fields $\overset{(i)}{g}_{AB}$ on S^{n-1} such that

$$(2.18) \quad \bar{g}_{AB} - r^2 s_{AB} = \sum_{i=4}^N r^i \overset{(i)}{g}_{AB} + O_{\ell-1}(r^\lambda).$$

Subsequently,

$$(2.19) \quad \bar{g}^{AB} - r^{-2} s^{AB} = \sum_{i=0}^{N-4} r^i \overset{(i)}{h}{}^{AB} + O_{\ell-1}(r^{\lambda-4}),$$

for some tensor fields $\overset{(i)}{h}{}^{AB} \in C^\infty(S^{n-1})$.

It is a consequence of the affine-parameterization condition and of the wave-gauge conditions that the function ν_0 solves the equation (see [6] for details)

$$(2.20) \quad \partial_1 \nu^0 = -\frac{1}{2} \tau \nu^0 + \frac{1}{2} \bar{g}^{AB} r s_{AB}.$$

Set

$$(2.21) \quad Y := 1 - \nu^0,$$

then (2.20) integrates to

$$(2.22) \quad Y(r) = r^{-\frac{n-1}{2}} \exp\left(\frac{1}{2} \int_0^r \psi(\rho) d\rho\right) \int_0^r \rho^{\frac{n-1}{2}} \exp\left(-\frac{1}{2} \int_0^\rho \psi(\chi) d\chi\right) F(\rho) d\rho,$$

where $\psi := -\tau + (n-1)/r$ and

$$(2.23) \quad F := \frac{1}{2}(\tau - \bar{g}^{AB} rs_{AB}) \equiv \frac{1}{2}\{(r^{-2}s^{AB} - \bar{g}^{AB})rs_{AB} - \psi\} = O(r),$$

We find successively

$$\begin{aligned} \int_0^r \psi(s) ds &= - \sum_{i=3}^{N-1} \frac{r^{i+1}}{i+1} \stackrel{(i)}{\tau} + O_{\ell-1}(r^\lambda), \\ F &= \sum_{i=1}^{N-3} r^i \stackrel{(i)}{F} + O_{\ell-1}(r^{\lambda-3}), \end{aligned}$$

for some $\stackrel{(i)}{F} \in C^\infty(S^{n-1})$. Closer inspection shows that $\stackrel{(1)}{F} = \stackrel{(2)}{F} = 0$. Integrating (2.22), we conclude that there exist functions $\stackrel{(i)}{\nu}_0 \in C^\infty(S^{n-1})$ such that

$$(2.24) \quad \nu_0 - 1 = \sum_{i=4}^{N-2} r^i \stackrel{(i)}{\nu}_0 + O_{\ell-1}(r^{\lambda-2}).$$

If $N < 6$ the sum from four to $N-2$ is understood as zero. One has a similar formula for $\nu^0 - 1$.

We pass now to a vector ξ_A , defined as

$$(2.25) \quad \xi_A := -2\nu^0 \partial_1 \nu_A + 4\nu^0 \nu_C \chi_A{}^C + \left(\bar{W}^0 - \frac{2}{r} \nu^0\right) \nu_A + \gamma_{AB} \gamma^{CD} (S_{CD}^B - \tilde{\Gamma}_{CD}^B).$$

Here the S_{BC}^A 's are the Christoffel symbols of the canonical metric s_{AB} on S^{n-1} , W^α is as in (2.6), the $\tilde{\Gamma}_{BC}^A$'s are the Christoffel symbols of the $(n-1)$ -dimensional metric \tilde{g}_{AB} , with associated covariant derivative operator $\tilde{\nabla}$, and $\nu^0 = 1/\nu_0$, while

$$\chi_A{}^B := \frac{1}{2} \overline{g^{AC} \partial_r g_{CB}}.$$

In terms of ξ_A the equation $R_{1A} = 0$ in wave-map gauge reads

$$(2.26) \quad -\frac{1}{2}(\partial_1 \xi_A + \tau \xi_A) + \tilde{\nabla}_B \chi_A{}^B - \frac{1}{2} \partial_A (\tau - \bar{W}_1 + 2\nu^0 \partial_1 \nu_0) = 0,$$

where $\bar{W}_1 = \nu_0 \bar{W}^0$. From what has been proved so far this last equation can be written as

$$(2.27) \quad \partial_1 \xi_A + \tau \xi_A = \sum_{i=1}^{N-3} r^i \stackrel{(i)}{f}_A + O_{\ell-2}(r^{\lambda-3}),$$

for some $\stackrel{(i)}{f}_A \in C^\infty(S^{n-1})$. The unique solution of this equation with the relevant behaviour at the origin satisfies

$$(2.28) \quad \xi_A = \sum_{i=2}^{N-2} r^i \stackrel{(i)}{\xi}_A + O_{\ell-2}(r^{\lambda-2}),$$

for some $\xi_A^{(i)} \in C^\infty(S^{n-1})$. Viewing (2.25) as an equation for ν_A , we obtain

$$(2.29) \quad \nu_A = \sum_{i=3}^{N-1} r^i \nu_A^{(i)} + O_{\ell-2}(r^{\lambda-1}),$$

for some $\nu_A^{(i)} \in C^\infty(S^{n-1})$.

We finally need to integrate the third wave-map-gauge constraint, $G_{10} = 0$, where $G_{\mu\nu}$ is the Einstein tensor:

$$(2.30) \quad \begin{aligned} & 2\partial_1^2 \bar{g}^{11} + 3\tau\partial_1 \bar{g}^{11} + (\partial_1\tau + \tau^2)\bar{g}^{11} \\ & + 2(\partial_1 + \tau)\bar{W}^1 + \tilde{R} - \frac{1}{2}\bar{g}^{AB}\xi_A\xi_B + \bar{g}^{AB}\tilde{\nabla}_A\xi_B = 0. \end{aligned}$$

We need the formula, for some functions $s^{(i)} \in C^\infty(S^{n-1})$,

$$(2.31) \quad \tilde{R} = \frac{(n-1)(n-2)}{r^2} + \sum_{i=0}^{N-4} r^i s^{(i)} + O_{\ell-3}(r^{\lambda-4}),$$

and the result, for a Minkowski target,

$$(2.32) \quad \begin{aligned} \bar{W}^1 &= \bar{W}^0 = -\frac{n-1}{r} + \sum_{i=3}^{N-3} r^i w^{(i)} + O_{\ell-1}(r^{\lambda-3}) \\ &= -\frac{n-1}{r} + O(r^3), \end{aligned}$$

with $w^{(i)} \in C^\infty(S^{n-1})$. Hence

$$(2.33) \quad \begin{aligned} 2(\partial_1 + \tau)\bar{W}^1 &= -2\frac{(n-1)(n-2)}{r^2} + \sum_{i=2}^{N-4} r^i v^{(i)} + O_{\ell-1}(r^{\lambda-4}) \\ &= -2\frac{(n-1)(n-2)}{r^2} + O(r^2), \end{aligned}$$

for some functions $v^{(i)} \in C^\infty(S^{n-1})$. A straightforward analysis of the remaining terms in (2.30) shows that solutions with the required asymptotics satisfy, for some functions $g^{(i)} \in C^\infty(S^{n-1})$,

$$(2.34) \quad \bar{g}^{11} = 1 + \sum_{i=2}^{N-2} r^i g^{(i)} + O_{\ell-3}(r^{\lambda-2}).$$

In order to apply Dossa's existence theory from [9] to the wave-map gauge reduced Einstein equations, we transform the metric to coordinates (y^μ) , regular near the tip of the light-cone, defined as

$$y^0 = x^1 - x^0, \quad y^i = x^1 \Theta^i(x^A),$$

where $\Theta^i(x^A) \in S^{n-1} \subset \mathbf{R}^n$. In these coordinates our wave-map-reduced Einstein equations become the usual harmonically-reduced equations, which have the right structure for the existence results in [9]. One finds that there exist functions $g_{\mu\nu}^{(i)} \in$

$C^\infty(S^{n-1})$ such that

$$\underline{\bar{g}_{\mu\nu}} - \underline{\eta_{\mu\nu}} = \sum_{i=2}^{N-2} r^i \underline{g_{\mu\nu}}^{(i)} + O_{\ell-3}(r^{\lambda-2}).$$

This proves the first part of Theorem 2.1.

Now, for general $\underline{s}_{AB}^{(i)}$'s the sum $\sum_{i=2}^{N-2} r^i \underline{g_{\mu\nu}}^{(i)}$ will *not* be the restriction of a polynomial to the light-cone. Assume, however, that this is the case. Then the remainder term in the difference $\underline{\bar{g}_{\mu\nu}} - \underline{\eta_{\mu\nu}}$ will be of the differentiability class $F^m(C_O^T)$ with $m > n/2 + 1$, as required by Dossa for existence [9], provided that

$$(2.35) \quad \lambda \geq 2m + 1 > n + 3.$$

The solution $\underline{g_{\mu\nu}} - \underline{\eta_{\mu\nu}}$ is then in Dossa's space $\tilde{F}^m(Y_O^T)$ for some $T > 0$, which embeds into $W^{m-n/2,\infty}(C_O^T) \subset W^{1,\infty}(C_O^T)$ leading to, for small $|t| + r$,

$$(2.36) \quad g_{\mu\nu} - \eta_{\mu\nu} = O_1(|t| + r), \quad \partial_\sigma g_{\mu\nu} = O(1).$$

In the coordinates (y^μ) the harmonicity vector can be calculated using the usual formula,

$$\bar{H}^\alpha = \frac{1}{\sqrt{|\det g_{\mu\nu}|}} \partial_\beta (\sqrt{|\det g_{\mu\nu}|} g^{\alpha\beta}),$$

which has bounded components in the (y^μ) coordinate system near the tip of the cone. The Bianchi identity (which, in its simplest version, requires a C^3 metric; this raises the differentiability threshold to $m > n/2 + 3$) together with the arguments in [6] show then that

$$\bar{H}^\mu = 0,$$

and thus the solution of the wave-map reduced Einstein equations, obtained from Dossa's theorem, is also a solution of the vacuum Einstein equations. \square

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